A Single Cluster Covering for Dodecagonal Quasiperiodic Ship Tiling

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Single cluster covering approach provides a plausible mechanism for the formation and stability of octagonal and decagonal quasiperiodic structures. For dodecagonal quasiperiodic patterns such a single cluster covering scheme is still unavailable. We demonstrate that the ship tiling, one of the dodecagonal quasiperiodic structures, can be completely covered by a single cluster. A deflation procedure is devised by assigning proper orientations to different tiles, and nine types of vertex configurations, if mirror patterns are considered to be identical, have been identified, which fulfill the closure condition under deflation and all result in a T-cluster centered at vertex.

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Quasicrystals are solids with long-range positional order and orientational order, but non-periodic translational order.[1,2] In the quasicrystals, 5-fold, 8-fold,[3,4] 10-fold[5] and 12-fold[6] rotational symmetries may appear, which are forbidden in the ordinary crystals. Unlike periodic crystals which can be constructed from one single unit cell, an ideal quasicrystal is often modeled as a tiling with two or more distinct building blocks. In the 2D case, the most renowned model is the Penrose tiling,[6] which comprises of fat and thin rhombuses subject to particular matching rules that constrain the ways how neighboring rhombuses join together edge-to-edge, cohering them into a fivefold-symmetric pattern.

Cluster covering is a new approach to describe the structure of quasicrystals.[7] In contrast to tiling models like the Penrose tiling, cluster covering demands only a single repeating unit, which is similar to the concept of unit cell in crystals. However, unlike in the models of periodic crystals, the repeating clusters in the cluster covering model must be allowed to partially overlap. Gummelt once proposed a decagonal covering scheme and showed that a quasiperiodic structure equivalent to the Penrose tiling could be constructed from decorated decagons.[7] Jeong and Steinhardt demonstrated that by maximizing the density of a chosen cluster comprising of several fat and thin rhombuses the Penrose tiling can be given rise.[8] Cluster covering can provide a plausible mechanism for the formation and stability of quasicrystals,[9] hence many other works have been devoted to cluster covering models of quasicrystals in the past years, and various single cluster covering models for decagonal and octagonal two-dimensional quasicrystals have been established.[10,11]

However, it is now still a challenge to extend the single cluster covering approach to dodecagonal quasiperiodic structures. Ben-Abraham and his coworkers, after proposing several “almost covering” patches, claimed conclusively in 2001 that “a complete covering of dodecagonal quasiperiodic structures requires two clusters”.[12] In Ref. [13], one of the current authors, in studying the structural properties of the dodecagonal quasiperiodic ship tiling,[13] noticed the existence of a Turtle-like cluster, which is dubbed as T-cluster and comprises of seven squares, twenty regular triangles and two 30°-rhombuses, as highlighted in dark gray in Fig. 1. At that time it came to us that this type of special cluster may make a perfect covering of the ship tiling. In this Letter, we present a simple proof that the ship tiling or the Stampfli–Gähler tiling[14] can be perfectly covered by using only T-clusters.

Stampfli[15] first constructed a dodecagonal quasiperiodic pattern from three distinct tiles: a square, a regular triangle and a 30°-rhombus. By using the cut-and-projection method Gähler[16] pushed this idea one step forward and obtained a structure similar to that of Stampfli, but the number of the rhombuses used is enormously reduced. Following the nomenclature of Ben-Abraham and his coworkers, this pattern was dubbed with the ship tiling.[13] Just like other quasiperiodic structures, the ship tiling is self-similar. This can be verified by applying the deflation procedure, a kind of self-similarity transformation,[17] to the ship tiling. The deflation procedure we devised, as illustrated in Fig. 2, replaces a square by five squares, sixteen regular triangles and four 30°-rhombuses (of the next generation), a regular triangle by three squares and seven regular triangles, and a 30°-rhombus by two squares, eight regular triangles and three 30°-rhombuses. More concisely, the deflation procedure can be specified by the deflation matrix

\[
T = \begin{pmatrix}
5 & 16 & 4 \\
3 & 7 & 0 \\
2 & 8 & 3
\end{pmatrix},
\]

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of which the two eigenvalues other than unity are $7 - 4\sqrt{3}$ and $7 + 4\sqrt{3}$, respectively. These two eigenvalues, mutually reciprocal, are the scaling factors of area; the corresponding scaling factor of length is $2 - \sqrt{3}$ or $2 + \sqrt{3}$. Remarkably, in so doing all the tiles must be labeled with a specific orientation to indicate the orientation of the deflation. Squares and 30°-rhombuses in Fig. 2 have been marked with arrows to show the orientation of the deflation. For the deflation of a regular triangle, as shown in Fig. 2(b), the full $D_3$ symmetry of the original triangle is preserved. The arrow to specify the orientation of the deflation configuration can thus point to any vertex of the original regular triangle along the bisection line through it. Thus for clarity, no arrows are added to the regular triangles. The specification of the orientation for the resulting tiles is an essential element in the deflation procedure, and it helps coordinate the deflation of edge-sharing units following the deflation rules. If the arrows are ignored, then there are certain tiles generated through the given deflation rules that may break the demanded 12-fold symmetry. In brief, Fig. 2 exemplifies the deflation scheme for the ship tiling of the dodecagonal quasicrystal structure. It should be mentioned that we obtained the deflation rules for the ship tiling, especially regarding the orientations of arrows on tiles, through analyzing the superimposition of ship tilings of 2–4 generations.

Using the deflation-inflation method, we can generate the dodecagonal ship tiling starting from a square, or a regular triangle, or a 30°-rhombus. The first step of the transformation is deflation as mentioned above, which changes a chosen tile into several smaller tiles of various types, scaled by a factor of $2 - \sqrt{3}$ with regard to the corresponding original ones. The second step is inflation, which rescales the new pattern by a factor $2 + \sqrt{3}$. By repeating this procedure again and again, we can produce a patch of dodecagonal ship tiling of an arbitrary size.

The ship tiling has a notable characteristic of edge sharing, that is, every edge of a square is always shared with a regular triangle, and so is of a 30°-rhombus, while a regular triangle may share one of its edges with a square, or a rhombus, or other regular triangles. This characteristic of edge sharing conforms to the aforementioned deflation rules, but we shall not go into the proof of this point here.

**Fig. 1.** The ship tiling of the first generation (bold, dashed lines) is superimposed upon that of the second generation (thin, solid lines). The highlighted region denotes a single T-cluster in the second generation.

**Fig. 2.** (Color online) Deflation rule for the ship tiling, as illustrated by the corresponding result (thin, black solid lines) of its application to the three original tiles (bold, red dashed lines): (a) a square, (b) an equilateral triangle, and (c) a 30°-rhombus. The arrows in the original tiles indicate the orientation of the resulting configurations. For clarity, the arrow on a regular triangle has been removed.

**Fig. 3.** The nine possible vertex configurations present in the ship tiling, with the orientation for further deflation of the tiles indicated by arrows.

On the basis of the deflation rules and the characteristic edge sharing, we find that if the arrowed directions are taken into account, there will be four-
teen possible vertex types in the ship tiling, but only nine if mirror patterns are considered to be identical, which are listed in Fig. 3. To facilitate the description below, we name each vertex type with the letter \( V \) followed by two numbers. For example, \( V5-3 \) refers to the third type of a vertex joining five tiles. The first seven vertex types in Fig. 3 can be easily obtained from schematic diagram of the deflation rules illustrated in Fig. 2. To find out the last two vertex configurations, i.e. \( V7-2 \) and \( V7-3 \), one can apply the deflation rules to the first seven vertex configurations. When \( V4-1 \) is concerned, the deflation procedure results in \( V7-3 \). When the other six configurations are concerned, the vertex configuration \( V7-2 \) can be obtained. From the results of applying the deflation procedure to \( V7-2 \) and \( V7-3 \), we can also find \( V7-2 \). That is to say, for the nine vertex types in Fig. 3, the closure condition is fulfilled.

A feature of the arrow directions of the two rhombuses in \( V7-1 \), \( V7-2 \) and \( V7-3 \) is noticeable. The arrows for the two \( 30^\circ \)-rhombuses in those three configurations always point to the same side of the straight line joining the bottom edges of the two \( 30^\circ \)-rhombuses. Just for this useful feature, once the orientation of the arrow of a \( 30^\circ \)-rhombus and the vertex to be shared are fixed, the position of the other \( 30^\circ \)-rhombus will be determined, which in turn can define the position and orientation of the corresponding T-cluster. Details will be given in the following.

\[\text{Fig. 4. (Color online) Deflation result (thin, black solid lines) of four typical vertex configurations (bold, red dashed lines): (a) \( V4 \) augmented with two triangles (bold, blue solid lines); (b) \( V5 \); (c) \( V6 \); (d) \( V7 \). For all the vertex configurations in Fig.3, the deflation gives rise to a T-cluster about the original vertex.}\]

Now we apply the deflation procedure to the nine vertex configurations, from the results we can find a striking fact that each vertex of the ship tiling of first generation corresponds to a T-cluster in the second-generation structure, if ignoring the decoration of arrows on tiles. Four cases are illustrated in Fig. 4, for instance, with T-clusters shaded in the figure. It is worth mentioning that T-clusters appearing in the results of deflation on \( V4-1 \) and \( V4-2 \) are incomplete. Yet, considering the characteristics of edge sharing mentioned above, if two supplementary regular triangles are added to them, as is always reasonable, a perfect T-cluster can be found for all occasions, see Fig. 4(a).

Now let us discuss the possibility of covering the ship tiling solely by the T-cluster. First, if three T-clusters in the second generation are centered at the three vertices of the parent regular triangle, obviously these three T-clusters completely cover the parent regular triangle. This is also true for the case of a \( 30^\circ \)-rhombus, if four T-clusters are centered at the four vertices of the parent \( 30^\circ \)-rhombus. In the case of a square, it is somehow a little complicated. There are four offspring \( 30^\circ \)-rhombuses within the deflation configuration for a parent square, and each of the four offspring \( 30^\circ \)-rhombuses has a well-specified orientation, see Fig. 2(a). Combining with the fact that each vertex of the first generation tiles corresponds to a T-cluster in the second generation centered at the vertex and the feature of the arrow directions of the two rhombuses in \( V7 \) mentioned above, we can easily construct the four offspring T-clusters corresponding to the four vertices of a square. Figure 5 shows an offspring T-cluster centered at the upper-left vertex of a square. It is easy to verify that the four offspring T-clusters corresponding to the four vertices of a square can make a perfect covering of the square.

\[\text{Fig. 5. (Color online) A T-cluster (in dark gray) in the second generation (thin, black solid lines) centered at one vertex of the original square (bold, red dashed lines). The four \( 30^\circ \)-rhombuses in the original square, when arrowed as in the figure, result from deflation in four T-clusters centered at each vertex, so oriented that they completely cover the original square.}\]

So far, we see that a regular triangle, as well as a \( 30^\circ \)-rhombus and a square, can be completely covered by three or four offspring T-clusters centered at the corresponding original vertices. Since the ship tiling
is constructed from squares, regular triangles and 30°-rhombuses, we can safely draw the conclusion that the ship tiling can be perfectly covered by a single T-cluster, in the sense that the ship tiling and the T-cluster are treated as purely geometrical objects. It is worth mentioning that there will be four distinct types of T-clusters if the arrows on the tiles are taken into account. When the T-clusters are to be decorated with atoms to make true quasicrystals, the arrows are then of essential importance.

In summary, we have demonstrated explicitly for the first time the deflation procedure for the ship tiling by assigning proper orientations to different tiles. In this process, nine possible types of vertex configurations existing in the ship tiling have been identified based on the deflation rules and the characteristic edge-sharing among tiles. By applying the deflation procedure to the nine vertex configurations, it is shown that ship tiling as one of the dodecagonal quasicrystal structures can be perfectly covered by only one single cluster, i.e. the T-cluster. This work is a step forward in constructing the covering for dodecagonal quasiperiodic structures in comparison to the previous ‘almost covering’ scheme, which may facilitate the study of physical properties of the dodecagonal quasicrystals.

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