Stress-driven buckling patterns in spheroidal core/shell structures

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Communicated by John W. Hutchinson, Harvard University, Cambridge, MA, October 20, 2008 (received for review July 5, 2008)

Many natural fruits and vegetables adopt an approximately spheroidal shape and are characterized by their distinct undulating topologies. We demonstrate that various global pattern features can be reproduced by anisotropic stress-driven buckles on spheroidal core/shell systems, which implies that the relevant mechanical forces might provide a template underpinning the topological conformation in some fruits and plants. Three dimensionless parameters, the ratio of effective size/thickness, the ratio of equatorial/polar radii, and the ratio of core/shell moduli, primarily govern the initiation and formation of the patterns. A distinct morphological feature occurs only when these parameters fall within certain ranges: In a prolate spheroid, reticular buckles take over longitudinal ridged patterns when one or more parameters become large. Our results demonstrate that some universal features of fruit/vegetable patterns (e.g., those observed in Korean melons, silk gourds, ribbed pumpkins, striped carnavales, and cantaloupes, etc.) may be related to the spontaneous buckling from mechanical perspectives, although the more complex biological or biochemical processes are involved at deep levels.

Spontaneous buckling of thin films on compliant substrates can achieve numerous highly ordered patterns due to mismatched deformation (1–6), which can be manipulated in different ways (1, 7–9). Buckling may also play an important role in the morphogenesis of some plant parts, including phyllotactic pattern in compressed tunica (10, 11), primordium initiation in sunflower capitulum (12), and Fibonacci patterns resembling those in some flowering cactus and pine cones (2), among others, in a way that is similar to the energy-minimizing buckling of a compressed shell on an elastic foundation (10).

Many natural fruits and vegetables can be approximated as spheroidal stiff exocarp (shell)/compliant sarcocarp (core) systems, which exhibit intriguing buckle-like profiles. For example, the Korean melon (yellow melon) and ridged gourd (or silk gourd, Cucumis melo) are distinguishable by 10 equidistant longitudinal ridges that run from stem to tip. Small pumpkins, acorn squashes, and carnival squashes often have ~10 uniformly spaced ribs, whereas the large pumpkins often have ~20 or more ridges. Similar undulating morphologies found in varieties of cucumis melons, gourds, striped carnavales, bell peppers, and other fruits and vegetables underpin their distinctive appearances. Although pattern formation in plants usually involves various complex biological and biochemical processes (11, 13, 14), such distinctive yet simple features make one wonder whether there exist other relatively simpler mechanisms contributing to the morphogenesis at the macroscopic scale, and the possibility of stress-driven buckling is explored in this study.

Consider a model spheroidal core/shell system where the shell is characterized by \((x^2 + y^2)/a^2 + z^2/b^2 = 1\) with equatorial radius \(a\), polar radius \(b\), and thickness \(t\) \((t \ll a)\); the system remains closed and bonded together in the due course of buckling. Both film (shell) and substrate (core) are assumed to be homogeneous, isotropic and elastic, with their Young’s moduli and Poisson’s ratios denoted as \(E_f\) and \(\nu_f\) and \(E_s\) and \(\nu_s\), respectively \((E_f > E_s)\). When the current model is applied to fruits and vegetables, the parameters \((E_f, E_s, t)\) should be regarded as the “effective” (or nominal) variables because the true properties of fruit/vegetable components may be inhomogeneous, anisotropic, and gradient. We also neglect the boundary effect (e.g., the constraint imposed by the stem and the navel at a polar end) and assume no external constraint during the growth. The simplified model system is used to reveal useful insights of the important parameters governing pattern formations.

The growth in plants can be analogously simulated (15) without any external guidance, and we assume that the unconstrained growth is isotropic and uniform within both film and substrate. Because the outer layer of plant meristems often expands faster than the inner one (12, 16), the growth expansion coefficient (rate) is larger in the shell than that of the core. The mismatch between the expansion of the stiff thin film and soft thick substrate causes the benign growth stress in the film compressive.

For a prolate spheroid \((k = b/a > 1)\) or oblate spheroid \((k < 1)\), the in-plane stress in the film is non-equiaxial (anisotropic) and increases with growth in the prebuckled state, where an interface pressure is acting on the film due to the expansion mismatch. Neglecting the shear stress at interface (9), the ratio between the prebuckling hoop \((\sigma_h)\) and longitudinal \((\sigma_l)\) growth stresses at any point in the spheroidal film is [see supporting information (SI) Text]

\[
\sigma_h/\sigma_l = 2 - \frac{1}{1 + \frac{x_0^2}{R^2} \left(\frac{1}{1 - \frac{1}{k^2}}\right)}
\]

where \(x_0\) is the radius of latitudinal circle and \(R = a^2/b\) is the radius of curvature at the pole. Apparently, \(\sigma_h/\sigma_l > 1\) in a prolate spheroid, whereas \(\sigma_h/\sigma_l < 1\) in an oblate spheroid; the stress ratio also depends on the size of spheroid. Denote \(\gamma = \max(\sigma_h/\sigma_l)\), the film starts to buckle when \(\gamma = \sigma_l\) and \(\sigma_l\) is a critical stress—above such critical stress level, various buckle patterns emerge depending on the geometrical and material constraints, and the undulations tend to relax the higher stress component. The spontaneous buckles are simulated by using the finite element method (FEM) (see SI Text).

Authors contributions: X.C. designed research; J.Y., I.S., and X.C. performed research; J.Y., I.S., and X.C. contributed new reagents/analytic tools; J.Y., Z.C., C.L., and X.C. analyzed data; and J.Y., Z.C., and X.C. wrote the paper.

The authors declare no conflict of interest.

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*We have shown that the typical boundary constraint by the stem/naval at the polar end of a spheroidal system does not affect the buckle pattern in a significant way.

This article contains supporting information online at www.pnas.org/cgi/content/full/0810443105/DCSupplemental.
Results and Discussion

Four effective dimensionless parameters, the normalized size $R/t = a^3/bt$, the shape factor $k = b/a$, the modulus ratio $E_f/E_s = E_f/(1 - v_f^2)/E_s/(1 - v_s^2)$, and the normalized growth stress $\sigma_g/\sigma_c$, primarily affect the stress-driven buckling patterns. In Fig. 1, for representative values of $R/t = 20$ and $E_f/E_s = 30$, the buckled patterns with $\sigma_f/\sigma_c$ and $k$ (transit from oblate to prolate shape) when $k = 1.0$, according to Eq. 1, $\sigma_f/\sigma_c = 1$, and thus, reticular buckle patterns are formed on spherical core/shell system when $R/t$ is not very large (17) and the buckle amplitude increases with $\sigma_f/\sigma_c$, as shown in Fig. 1. In spherical systems with large $R/t$, labyrinth undulations appear—the buckling mechanisms of spherical film/substrate systems have been studied in ref. 17. Note that the stress field is also isotropic $(\sigma_f/\sigma_c = 1)$ if the substrate is planar without further manipulation of system properties (e.g., anisotropic growth), and the resulting buckling patterns have no directional preference; the principles of pattern formations on planar substrates (under both isotropic and anisotropic stresses) have been discussed extensively in earlier works, e.g., refs. 8, 9, and 18–20. In this present article with the spheroidal substrate, the stress is anisotropic when $k \neq 1$, which leads to distinct buckling mechanisms and more variety of undulation morphologies than those on a spherical substrate (17), and they are explored in detail as well as their implications in fruit morphogenesis.

In the case of slight deviation from spherical shape, e.g., setting $k = 0.9$ and 1.1, ordered latitudinal and pumpkin-like ribbed patterns are formed at the onset of buckling, respectively, which then transit to triangular patterns at higher stresses. With the more evidently oblate or prolate spheroids, the buckling pattern persists with film growth. In particular, when $k$ is larger than $\sim 1.3$, the hoop/longitudinal growth stress ratio $\sigma_f/\sigma_c$ becomes increasingly $\sim 1$ and reaches its maximum at the equator; consequently, equidistant longitudinal buckle ribs are first formed near the equator and extend to the poles as $\sigma_f/\sigma_c$ is increased.

Because most fruits and vegetables with undulating morphology are prolate (in an approximate sense), in the following we focus on the buckling in prolate spheroids (especially those with $k \geq 1.3$). With reference to Eq. 1, if $R/t$ is very large, $\sigma_f/\sigma_c$ decreases to approach the value of 1.0, consequently circumferential buckles may emerge (Fig. S1). At lower values of $R/t$ ($R/t \leq 30$), the stress anisotropy is higher and the ribbed morphology remains quite robust with continued moderate level of shell growth, which can be confirmed when both $E_f/E_s$ and $k$ are varied in a large range. Fig. S1 and Fig. S2 if the growth strain is too large, the ribbed morphologies may become unstable (19). Owing to such a relative stability, we will explore subsequently how the initial buckle pattern ($\sigma_f/\sigma_c \sim 1.05$) varies with the 3 dimensionless ratios $(k, R/t, E_f/E_s)$.

In the first buckling map (Fig. 2), $E_f/E_s = 30$ while $k$ and $R/t$ are varied. The solid line is a rough boundary to distinguish the geometrical parameter space of longitudinal ridged patterns from that of reticular ones. When $k$ is fixed, with the increase of $R/t$ (i.e., the larger $a/t$), the number of longitudinal ribs grows rapidly in accompany with the occurrence of circumferential waves, and the buckle finally transits to a reticular pattern at large $R/t$. Governed by the competition between $k$ and $R/t$ on growth stress (Eq. (1)), the longitudinal ridged buckle takes over at larger $k$ and/or smaller $R/t$; otherwise the addition of circumferential waves makes the buckles reticular. At constant $R/t$, with the increase of $k$ the ridged buckle dominates and the ridge number also increases primarily due to the increase of $a/t$. In such longitudinal ridged patterns the number of ribs is essentially only a function of $a/t$ and $E_f/E_s$, as will be discussed below.

For a representative $k = 1.3$, the buckling map in Fig. 3 shows that at a fixed geometrical factor ($R/t$), the wave length of the buckled ridges increases with the elastic mismatch. At small $E_f/E_s$ and large $R/t$, reticular pattern ensues; otherwise, ribs prevail which may extend to the polar regions as $R/t$ increases. Near the boundary (solid line), one could identify the upper limits of the ribs in the stable longitudinal ridged configuration. When $R/t$ is relatively small, the wave number appears to be quite stable against a large variation of $E_f/E_s$.

At large $k$, the prolate spheroid may be approximated as a long cylinder of radius $a$. The buckle wave number can be estimated as:

$$R_{cr} \approx \left( \frac{a}{l} \right)^2 \left( \frac{12E_f}{E_s} \right)^{1/2}$$

[2]
where \( E_s = E_u/(1 + v_s)(1 - 2v_s) \) (see SI Text). Even at a relatively small \( k \), \( k = 1.3 \) say, the wave number predicted by Eq. 2 is reasonably close to that obtained from FEM simulations of prolate spheroids (Fig. S3); the small difference is due to the geometrical constraints at capped poles and also to the system instability at large \( R/t \). Hence, the characteristics of longitudinal ridged pattern is dominated by \( a/t \) (substrate curvature) and \( E_s/E_u \) (film/substrate modulus mismatch) and more ribs are possible with larger \( a/t \) and/or smaller \( E_s/E_u \), as validated from Figs. 2–3.

Despite the complicated cellular processes involved during the growth of some fruits and plants, mechanical buckling may reproduce similar ribbed and reticular morphologies for prolate spheroidal core/shell structures (Fig. 4). The results demonstrate that a distinct profile is possible only when the 3 geometrical and material parameters fall within a certain range. For example, examination of the Korean melon shows that its effective \( E_s/E_u \) is \( \sim 15–18 \); and \( k \) is \( \sim 1.2–1.4 \), which may justify its 10 equidistant longitudinal ridges (see Fig. 2); in addition, to maintain such a profile, the effective \( E_s/E_u \) may vary between \( \sim 20 \) and 40 (see Fig. 3). Similarly, if the distinct patterns are related to buckling, for ridged gourd whose \( k \approx 5–6 \), its effective \( R/t \) and \( E_s/E_u \) should be \( \sim 4–5 \) and 20–30, respectively; for striped cavern tomatoes with 4–6 ridged buckles, the effective \( R/t \) is \( \sim 5 \); and for reticular patterns to prevail, cantaloupes (\( k \approx 1.1–1.2 \)) should have a large \( R/t \) and a low \( E_s/E_u \). A resemblers of pumpkins may be obtained by modifying the spheroid model (Fig. S4); because the wave number is related to the normalized equatorial radius \((a/t)\). Fig. 4 shows that in a small pumpkin simulant, 10 ridges may emerge, and when \( a/t \) grows larger, 20 or more ribs are possible. In all cases, the ridged patterns are also robust when the effective mismatch \( E_s/E_u \) varies within a moderate range (Fig. 3); indeed, moderate variations of material properties are reasonable during fruit/vegetable growth (21) and despite the scatter of properties in individual fruits/vegetables, certain species (e.g., Korean melon) still exhibit a distinct topology.

There is no doubt that biological and biochemical processes also play a very important role in regulating the plant pattern formation. In leaf morphology for example, the mutation of CIN gene found in Antirrhinum exhibits excess growth in marginal regions (22), which makes the edges in compression and turns the normally flat leaves into wavy surfaces with buckled margins. Similar studies on Arabidopsis (23) and eggplant leaf (24) suggest a role for mechanical force in the regulation of plant morphology through its interaction with biological processes. The phenomenological study of stress-driven buckling in a spheroidal core/ shell system may suggest a similar regulatory pathway: With the excessive growth of the shell-like exocarp (through certain biological processes), buckling occurs due to mechanical forces where the initiation locations and characteristics (e.g., the number of ribs) are roughly controlled by the aforementioned mechanical principles. Meanwhile, the effective material properties relevant to buckling modes are based on cellular properties of the exocarp and sarcocarp. During the buckling process, the mechanical forces may also confine and interact with certain biological processes such as cell growth and cell differentiation (11, 14) to help stabilize the global pattern features. In other words, the morphology of some plants/fruits may result from a concerted effort of the mechanical and biological processes.

We notice that the dehydration process or a fruit or vegetable can be free from the biological factors that could have been involved in its growth; therefore the surface pattern of a dehydrated fruit (or vegetable) is dominated by buckling. In a dried mini tomato (Fig. S5), the longitudinal ridges are analogous to that shown in Fig. 3 and the wave numbers are also associated with \((k, R/t, \text{and } E_s/E_u)\) (the latter 2 factors are of more weight). In addition, experiments were carried out on a SiO\(_2\) shell/Ag core microsystem, and the observed buckle patterns (Fig. S6) qualitatively agree with simulation (see SI Text); such results also demonstrate the potential applications in micro- and nanofabrications via controlled self-assembly on curved/closed substrate surfaces.

**Conclusion**

Although Charles Darwin remarked that the explanation of plant pattern formation could “drive the sanest man mad” (25), anisotropic stress-driven buckling in spheroidal core/shell system, as we...
have shown here, could, to some extent, resemble the global pattern features formed in the appearance of quite many kinds of fruits and vegetables, and each distinct pattern corresponds to certain ranges of the three dimensionless material and geometrical parameters. Mechanical buckling might have provided a template for the morphology development in some fruits/plants that may confine and interact with certain biological processes such as cell growth and cell differentiation. Note that we employ the simplest spheroid model, and such a model could be refined to include the end/ boundary (stem) constraints, anisotropic growth, and more sophisticated material/structural details, among others. Nevertheless, the encouraging results obtained through the self-assembled patterns of the model system show that the initiation and macroscopic topological characteristics of these patterns may be connected to mechanical principles, and the morphology of plants/fruits may be manipulated by the interaction between mechanical forces and biological processes.

Materials and Methods
To establish the model system in Cartesian coordinates, the spheroidal surface \((x^2+y^2)a^2+z^2b^2-1=0\) is created by rotating the ellipse, \(x^2/a^2+z^2/b^2=1\), about the z axis (vertical axis). The film thickness, \(t\), is assumed to be much smaller than both \(a\) and \(b\), and the film subtends a complete cover to the substrate. The film (in analogue to the fruit/vegetable shell) remains bonded to the substrate (core) throughout deformation. The growth is assumed to be isotropic within both film and substrate, but the unconstrained growth coefficient (expansion rate) is larger in the film. The mechanics of buckling is regulated by the minimization of system potential energy (because the film is bonded to the substrate, the substrate also deforms). Buckling simulation has been performed by using the finite element method (FEM). More details can be found in SI Text.

ACKNOWLEDGMENTS. This work was supported by National Science Foundation Grant CMMI-CAREER-0643726, National Science Foundation of China Grants 60621091 and 50772100, and the Civil Space Exploration Program of China.

Supporting Information

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SI Text

Spheroid Model. To establish the model system in Cartesian coordinates, the spheroidal surface \((x^2 + y^2)/a^2 + z^2/b^2 = 1\) is created by rotating the ellipse, \(x^2/a^2 + y^2/b^2 = 1\), about the z axis (vertical axis). The film thickness, \(t_f\), is assumed to be much smaller than both \(a\) and \(b\), and the film subtends a complete cover to the substrate. The film (in analogue to the fruit/vegetable shell) remains bonded to the substrate (core) throughout deformation.

The growth stress may be effectively modeled and simulated in a phenomenological way (1) without external guidance. The growth is assumed to be isotropic within both film and substrate but the unconstrained growth coefficient (expansion rate) is larger in the film. In essence, the more prominent growth in the film makes the outer layer in compression, and the film surface undulates to a certain pattern so as to partially relieve the anisotropic compressive stress—in a mechanical regime, it means buckling. Similarly, during the dehydrating process the core (flesh) of a fruit or vegetable loses more water content than the shell (skin), and the skin also subjects to similar compression, thus it buckles. The mechanics of buckling is regulated by the minimization of system potential energy (because the film is bonded to the substrate, the substrate also deforms).

Computation Method. Buckling simulation has been performed by using the finite element method (FEM) with software ABAQUS (2). The substrate is meshed by over 700,000 hexahedron elements and the film is represented by over 60,000 4-node general-purpose shell elements with reduced integration and accounting for large rotation. Larger spheroids have more elements. In all cases, the mesh density is validated from mesh convergence studies.

For a system with given \(k\), \(R/t\), \(E_f/E_s\), the growth strain is increased from zero. That is, we vary \(\sigma_f/\sigma_c\) gradually from 0.0 to above 1.0 and record the undulation morphologies when \(\sigma_f/\sigma_c\) > 1.0. When a system parameter (\(k\), \(R/t\), or \(E_f/E_s\)) is changed, the strain is reset to zero and then \(\sigma_f/\sigma_c\) is increased gradually (from 0.0 to above 1.0) in the new simulation. Note that to initiate buckling, a very small random geometrical perturbation is introduced to the system in the beginning. Such an initial numerical “defect” has very little effect on the buckled pattern at \(\sigma_f/\sigma_c\) = 1.0, and it has essentially no effect on the results of the more “stabilized” patterns at \(\sigma_f/\sigma_c\) ~ 1.05, which are the situations handled in the current article.

The characteristics of buckling patterns, in particular the wave number of longitudinal ridges, is marked in each simulated buckled configuration in Figs. 1–4 and Figs. S1 and S2.

Supplementary Simulation Results. In addition to the several buckling maps shown in the text, in this section we provide more evidences to show that for prolate spheroids, the undulation pattern is relatively stable with \(\sigma_f/\sigma_c\) (as long as it is not too large), which would allow one to concentrate on the initial buckling shape (Figs. 2 and 3 in the main text).

In general, at the onset of buckling, in a prolate spheroid longitudinal buckles prevail, whereas in an oblute one, circumferential waves are advantageous for relieving the larger anisotropic stress component. When the hoop growth stress is close to longitudinal growth stress, both longitudinal and circumferential waves emerge and a reticular pattern appears. The buckle amplitude increases with \(\sigma_f/\sigma_c\) and if the stress is very high, the buckle pattern changes so as to relieve the strain energy more effectively, and in some cases (especially at large \(R/t\), smaller \(k\), and/or smaller \(E_f/E_s\)) the reticular buckles take over.

Fig. S1 shows that for a spheroid with \(k = 1.3\) and \(E_f/E_s = 30\), longitudinal ribs are preferred at smaller \(R/t\) and reticular patterns are favored at larger \(R/t\), and these patterns are quite stable as \(\sigma_f/\sigma_c\) varies within a moderate range above the critical buckling stress (since the stress anisotropy is high in a distinct spheroid). When the growth stress is too high, however, circumferential waves may emerge and the longitudinal ribs may transit to a reticular pattern and/or eventually to a reticular pattern (3)—such instability and pattern transition at higher film stress is also consistent with the anisotropic stress-driven buckling of thin film on a planar substrate (4). Combined with the information in Fig. S2, as long as \(k \geq 1.3\), \(E_f/E_s\) is not too small, and \(R/t\) is not too large, for a wide range of material and geometrical parameters, the buckled patterns are quite stable with moderate film growth, and thus we can focus on the initial buckle patterns which depend on \((k, R/t, E_f/E_s)\).

Fig. S1 and Fig. S2 also show that when \(k\) (or \(R/t\)) is fixed, the number of longitudinal ridges increases with \(R/t\) (or \(k\)), primarily due to the growing \(a/t\). The wavelength of the ridges also decreases with \(E_f/E_s\) (Fig. S2). These features can be related to the analytical solution below.

Analytical Results. Buckles on planar substrates can be manipulated by anisotropic stress (5, 6). Upon expansion, a pressure emerges at the interface between film and substrate due to the mismatch of growth. When the shear stress is neglected at the interface, one could study the characteristics of the onset of buckling when the spheroid is under pressure so as to obtain some useful insights.

Consider a film with closed curved surface (with thickness \(t\)) and under external pressure \(p\), at any point whose principal radii in the longitudinal and hoop directions are \(R_1\) and \(R_2\), respectively. The prebuckling compressive longitudinal stress \(\sigma_l\) and hoop stress \(\sigma_h\) can be expressed as \(\sigma_l = -pR_2/2t\) and \(\sigma_h = \sigma_l(2 - R_2/R_1)\), respectively (7). For the spheroid under consideration, the anisotropic stress ratio \(\sigma_h/\sigma_l\) is then:

\[
\frac{\sigma_h}{\sigma_l} = 2 - \frac{R_2}{R_1} = 2 - \frac{1}{1 + \frac{x_0}{R^2}(1 - \frac{k^2}{k^2})} \tag{1}
\]

where \(x_0\) is the radius of latitudinal circle. \(\sigma_h/\sigma_l\) is close to or smaller than 1 when \(R/t\) is very large or when \(k\) is small (e.g., in near-spherical or oblative spheroids), which would favor the initiation of circumferential buckles. Otherwise, the longitudinal ridges are preferred especially in prolate spheroids with smaller \(R/t\).

Another important parameter to characterize the buckling patterns is the critical buckling wavelength (or wave number at onset of buckling), which can be obtained from the minimum potential energy of film-substrate systems. As a first-order approximation, we approximate a prolate spheroid with large \(k\) as a cylinder of radius \(a\) and height \(l\) in the \(z\) direction. The plane strain problem of buckling of an elastic ring attached to an elastic foundation (upon external pressure) can be solved analytically. Through the minimization of the potential energy of the ring–foundation system, the critical buckling wave number \(n_{cr}\), which corresponds to the onset of buckle, can be derived from the critical buckling load solution proposed by Brush and Almroth (8) (assuming the wave number \(n\) is continuous and larger than 2)
with $E_r = E_f/(1 + v_f)(1 - 2v_f)$. The critical wavelength is $\lambda_{cr(1.1) to 2.0). We have validated this equation by comparing with plane strain FEM simulations on the same ring-foundation system.*

In Fig. S3, we compare the critical buckling wave number predicted by the simple ring model (Eq. 2) with that measured from the FEM simulations on prolate spheroids. Theoretically, the ring model is valid only when $k$ is very large. In Fig. S3, the shape index $k$ is fixed as 1.3. For different material modulus ratios, the ring model agrees well with FEM results of spheroids when $a/t$ is less than ~20. The error is in part due to the geometrical constraints at capped poles (when $k$ is small), and the error increases at larger $a/t$ with the formation of circumferential ridges in the spheroid. Nevertheless, the percentage error is not very large in all cases in Fig. S3 and considering the fact that $k$ is only 1.3, such agreement is quite remarkable. Moreover, the good agreement also holds for other $k$ values examined in the form 1.1 to 2.0.

In essence, Eq. 2 dictates the variation trend of the ridge number with $a/t$ and $E_r/E_f$, which is qualitatively validated in Figs. 1–3 and Figs. S1 and S2 and discussed in the text. The dependence of buckling wavelength on $a/t$ shows that the buckling of the film is very sensitive to the curvature of the underlying substrate (for a stiff film on a planar substrate subjecting to in-plane uniaxial compression (9) or equi-biaxial compression (10), the critical buckling wavelength depends only on $t$ and $E_r/E_f$).

Possible Connection with Fruit and Vegetable Patterns. The buckling patterns of several prolate spheroids are shown in Fig. 4 in the text, which qualitatively reproduce the overall shape features of several fruits and vegetables. The effective parameters used in the simulation examples shown in Fig. 4 are: ($R/t = 5, k = 1.3, E_r/E_f = 100$) for striped cavern tomato; ($R/t = 15, k = 1.3, E_r/E_f = 30$) for Korean melon; ($R/t = 4, k = 5, E_r/E_f = 30$) for ridged (silk) gourd; and ($R/t = 75, k = 1.3, E_r/E_f = 5$) for cantaloupe. The parameters for obtaining Fig. 4 remind us again that these numbers used in individual examples are meant only to be effective ones, and a typical buckle feature (e.g., 10 ridges) is stable against a moderate range of parameters.

Pumpkin, however, does not conform very well to the oblate spheroid model because the curvature on the side (near the equator) is relatively small. Instead, we propose to create a pumpkin simulant in the following way (Fig. S4). Starting from an ellipse in the $x$-$z$ plane, we cut the ellipse in half along the $z$ axis and then move the 2 parts away from each other along the positive and negative directions of the $x$ axis, respectively. Next, 2 straight horizontal lines are used to connect the tips of the 2 half ellipses. The rotation of the new shape in Fig. S4 around the $z$ axis would create an oblate-like structure except that the curvature on the side is relatively small; the ratio of the polar/equatorial radii of the simulant may be denoted as $k'$.

By following the same simulation procedure, the buckled shape of a small pumpkin (or acorn squash) simulant ($a/t = 20, k' = 0.8, E_r/E_f = 20$) shows 10 longitudinal ridges, whereas a large pumpkin simulant ($a/t = 50, k' = 0.6, E_r/E_f = 20$) exhibits 20 ribs (Fig. 4); in this case the wave numbers also agree well with that predicted by Eq. 2. Note that depending on the ranges of geometrical and material parameters, theoretically, any number of longitudinal ribs is possible in the pumpkin simulant, and more in the larger pumpkins. However, in reality, if a young pumpkin shows 10 ridges, it may be more straightforward for the ridges, following the growth, to split into two (to relieve the compressive stress) such that the ripe pumpkins could have ~20 buckles. In such a scenario, the buckled morphology of a young pumpkin may serve as a template for the subsequent morphological transitions as it grows. Bifurcation is a quite effective route leading to more complex patterns in many dynamical systems.

Dehydration of Fresh Fruits. A mini tomato ($a = 1$ cm and $k = 1.4$) was dried at 50 °C for 25 h, and since the flesh (core) shrinks faster than the skin (shell), the system buckles. Fig. S5 shows that the observed buckles are mostly longitudinal although there are also transverse (circumferential) waves. This indicates that the effective parameters are $R/t \approx 40–65$ and $E_r/E_f \approx 20–40$. Such dehydration experiments may be carried out on grapes, jujubes, etc. (for those with larger $k$), and similar longitudinal rib-dominated patterns are observed. When $R/t$ gets large while $k$ is small in a prolate fruit, e.g., an eggplant or a prune, labyrinth patterns prevail when dried.

Stressed Patterns on Spheroidal SiO$_2$ Shell/Ag Core Structure. Stress engineering on inorganic spheroids was carried out on the microstructures which were prepared by coevaporating high-purity SiO and Ag$_2$O powders at a high temperature ~1,535 K. The evaporation was done in an Al$_2$O$_3$ crucible enclosed in a reaction chamber that was at first evacuated and then filled with a gas mixture of 90% Ar + 10% H$_2$ to a pressure of 3 x 10$^4$ Pa. Liquid droplets of distinct Ag core/SiO$_2$ shell structure, due to the low solubility of SiO$_2$ in Ag, in a typical size of ~15 μm were collected by a substrate hanging over the evaporation source. After maintaining the system at given temperatures for ~15 min, the system was let cool down at a cooling rate of about 4–5 K/s to induce stress in the system. Those core/shell structures just arriving at the substrate, now may turn wrinkled, are available for later ex situ observation by using a scanning electron microscope (FEI, SERION) operated at 5.0 KeV. The long depth of field of an electron microscope enables a clear imaging of structural details on a curved surface, hence all of the patterns on the core/shell structures can be registered.

Fig. S6 shows a SEM photo of buckle patterns observed in a prolate spheroid, where $k = 1.15$, $a/t \approx 55$, and $E_r/E_f \approx 5$, respectively. The corresponding FEM simulation also shows the longitudinal ridge-dominated pattern that is interfaced with circumferential waves. Again, the theoretical prediction and the experimental observation agree well with each other, and the buckling mechanism may be applied to spheroidal core/shell structures at various microscopic and macroscopic scales.

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Fig. S1. Deformation map of a prolate spheroid as both $R/t$ and film stress are varied (with $k = 1.3$ and $E_s/E_0 = 30$). In all buckled morphologies in Figs. 1–3 and Figs. S1 and S2, the dark (red) areas mean that within such concave region the radial displacement magnitude is smaller than a certain value; all snapshots of the morphology are rescaled to fit into the map. For ridged morphology the number of buckles is shown on the top corner of each pattern.
Fig. S2. Deformation map of a prolate spheroid as film stress is varied. On the left, \( \frac{E_f}{E_s} \) is varied while \( k = 1.3 \) and \( R/t = 15 \); on the right, \( k \) is varied while \( \frac{E_f}{E_s} = 30 \) and \( R/t = 15 \).
Fig. S3. Comparison between FEM simulation and analytical solution Eq. 2. The critical buckling wave number $n_{cr}$ as a function of $a/t$, for 5 different modulus mismatch ratios and with $k = 1.3$. 
Fig. S4. Illustration of the model of the pumpkin simulant.
Fig. S5. Comparison between simulation (Right) with experiment (Left) on the buckling patterns in a dried mini tomato.
Fig. S6. Comparison between simulation (Right) with experiment (Left) on the buckling patterns in a stressed Ag core/SiO$_2$ shell system.